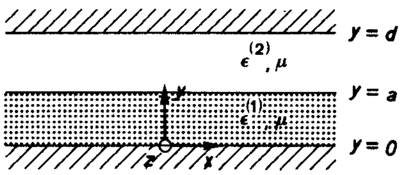

Fig. 3. Two-dielectric transmission line considered in the example of Section VII.

TABLE I

$\chi = 2 \quad b_1 = 0.241840\text{E } 02 \quad b_3 = 0.736682\text{ E } 02 \quad b_5 = 0.964940\text{ E } 05$

f (GHz)	β (m^{-1})	R_1 (m^{-1})	R_3 (m^{-1})	R_5 (m^{-1})
0.5	12.093	0.001	0.000	0.000
1	24.191	0.007	0.000	0.000
2	48.427	0.059	0.000	0.000
3	72.753	0.201	0.002	0.000
4	97.217	0.481	0.010	0.000
5	121.870	0.950	0.029	−0.001
6	146.766	1.662	0.071	−0.004
7	171.963	2.676	0.149	−0.013
8	197.523	4.051	0.279	−0.037
9	223.504	5.848	0.478	−0.092
10	249.963	8.124	0.757	−0.208

Note: $R_1 = \beta - b_1\Omega$; $R_3 = \beta - (b_1\Omega + b_3\Omega^3)$; $R_5 = \beta - (b_1\Omega + b_3\Omega^3 + b_5\Omega^5)$.

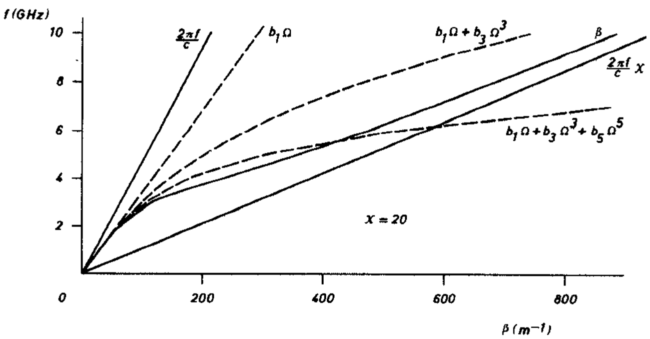

Fig. 4. Power series approximations to the phase constant of the line represented in Fig. 3.

quency; this is a TM mode with components $H_x = H$, E_y, and E_z. A standard procedure enables us to obtain the dispersion equation:

$$k^{(1)} \frac{\tan\,(k^{(1)}a)}{\epsilon^{(1)}} = k^{(2)} \frac{\tan\,[k^{(2)}(d-a)]}{\epsilon^{(2)}}$$

where superscript (1) refers to the dielectric layer $0 < y < a$ and superscript (2) to the dielectric layer $a < y < d$ and for each layer

$$k^2 = \omega^2 \epsilon \mu_0 - \beta^2.$$

Next the first three coefficients of the expansion for β, that is b_1, b_3, and b_5, were calculated, the last one with the sole purpose of assessing the speed of convergence.

The coefficients were obtained by the method developed in Sections V and VI, using the zero-order coefficient H_0 as the scale constant and imposing a frequency-independent current in the conductor $y = 0$.

The numerical computation was carried out for two cases corresponding to the following parameters:

$$d = 1 \text{ cm}$$

$$a = 0.5 \text{ cm}$$

$$\epsilon^{(2)} = \text{vacuum permittivity}$$

$$\chi = \epsilon^{(1)}/\epsilon^{(2)} = 2; 20.$$

The normalization frequency $f_0 = \omega_0/2\pi$ was taken to be 1 GHz.

For $\epsilon^{(1)}/\epsilon^{(2)} = 2$ the results are given in Table I; examination of this table shows that the error R_5 is less than 1/1000 of β within the frequency range considered.

For $\epsilon^{(1)}/\epsilon^{(2)} = 20$ the results are shown in Fig. 4; it is seen that the accuracy of the approximation degrades very quickly from the point where R_5 changes from positive to negative.

VIII. CONCLUSIONS

In the preceding sections it has been shown that for a transmission line with two conductors and a dielectric medium consisting of various homogeneous regions it is possible to expand all field functions as a power series of the frequency.

The main interest of this expansion appears to be the possibility of estimating an upper limit to the frequency band in which the dispersion does not exceed a specified value.

In this short paper the analysis has been confined to general aspects of the proposed expansion. The problem of computing the higher order terms for transmission lines of practical interest has not been considered.

ACKNOWLEDGMENT

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Useful Matrix Chain Parameter Identities for the Analysis of Multiconductor Transmission Lines

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***Abstract*—By utilizing state variable theory, certain useful matrix identities involving submatrices of the chain parameter matrix for a multiconductor transmission line are shown. These identities are extensions of familiar properties associated with two-conductor lines to multiconductor lines and are used to formulate the complete solution for the terminal currents when the line is terminated by linear networks. The identities allow a simplified solution for these currents and reduce numerous redundant time-consuming matrix multiplications. In addition, the correspondence between familiar terms for the two-conductor case and the multiconductor case is shown.**

I. INTRODUCTION

The subject of coupled transmission lines arises in the study of many microwave related structures. Transmission lines in a homogeneous medium occur in the study of strip lines whereas applica-

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tions involving transmission lines in an inhomogeneous medium occur in the study of microstrip lines. For transmission lines in a homogeneous medium, the principal mode of propagation is the TEM mode whereas for an inhomogeneous medium, the predominant mode of propagation is taken to be the "quasi-TEM" mode. For either mode of propagation, the per unit length distributed parameters in the transmission line representation are computed by assuming that the electric and magnetic field intensity vectors are transverse to the direction of propagation. Therefore, at each point along the line, these vectors satisfy static distributions [1].

The matrix chain parameter (or *ABCD* parameter) representation of the transmission line is often used in characterizing the line for the TEM (or "quasi-TEM") mode of propagation as a multiport network. The purpose of this short paper is to extend certain well-known identities involving elements of the chain parameter matrix for the two-conductor line to the multiconductor case. In addition, a convenient matrix formulation is shown which allows an efficient numerical solution for the terminal currents when multiport networks are connected by the line.

An $(n+1)$ conductor, uniform transmission line is considered with the $(n+1)$st conductor (usually an infinite ground plane or overall shield) designated as the reference conductor. The dielectric medium surrounding the conductors is assumed to be linear and isotropic but may be inhomogeneous. The line is considered to be uniform in that all $(n+1)$ conductors have uniform cross sections along their lengths, are parallel to each other and the x direction, and, in the case of an inhomogeneous medium, the characteristics of the medium exhibit no cross-sectional variation with x and are therefore independent of x. The line is of total length $\mathcal{L}$. For sinusoidal excitation, the voltage of the ith conductor with respect to the reference conductor is denoted by $\mathcal{V}_i(x,t) = V_i(x)e^{j\omega t}$ and the current associated with the ith conductor and directed in the positive x direction is denoted by $\mathcal{I}_i(x,t) = I_i(x)e^{j\omega t}$. $V_i(x)$ and $I_i(x)$ are complex valued and functions of x only with $i = 1,\cdots,n$. The transmission line is described for the TEM (or "quasi-TEM") mode of propagation and the sinusoidal steady state by the following set of $2n$-coupled first-order complex ordinary differential equations [1]:

$$\begin{bmatrix} \dot{\boldsymbol{V}}(x) \\ \dot{\boldsymbol{I}}(x) \end{bmatrix} = \begin{bmatrix} {}_n\boldsymbol{0}_n & -\boldsymbol{Z} \\ -\boldsymbol{Y} & {}_n\boldsymbol{0}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{V}(x) \\ \boldsymbol{I}(x) \end{bmatrix}. \tag{1}$$

A matrix $\boldsymbol{M}$ with m rows and p columns is of order $m \times p$. The element in the ith row and jth column of $\boldsymbol{M}$ is designated by $[\boldsymbol{M}]_{ij}$ with $i = 1,\cdots,m$ and $j = 1,\cdots,p$. The $n \times 1$ complex-valued vectors $\boldsymbol{V}(x)$ and $\boldsymbol{I}(x)$ have entries $[\boldsymbol{V}(x)]_i = V_i(x)$ and $[\boldsymbol{I}(x)]_i = I_i(x)$ in the ith rows, and the first derivative of a vector $\boldsymbol{V}(x)$ with respect to x is denoted by $\dot{\boldsymbol{V}}(x)$. An $m \times p$ zero matrix with zeros in every position is denoted by ${}_m\boldsymbol{0}_p$, i.e., $[{}_m\boldsymbol{0}_p]_{ij} = 0$ for $i = 1,\cdots,m$ and $j = 1,\cdots,p$.

The $n \times n$ complex symmetric matrices $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are the per unit length impedance and admittance matrices, respectively. These matrices are independent of x, since the line is assumed to be uniform, and are separable as

$$\boldsymbol{Z} = \boldsymbol{R}_c + j\omega\boldsymbol{L}_c + j\omega\boldsymbol{L} \tag{2a}$$

$$\boldsymbol{Y} = \boldsymbol{G} + j\omega\boldsymbol{C} \tag{2b}$$

where the $n \times n$ real matrices $\boldsymbol{R}_c$, $\boldsymbol{L}_c$, $\boldsymbol{L}$, $\boldsymbol{G}$, $\boldsymbol{C}$ are the per unit length conductor resistance, conductor internal inductance, external inductance, conductance, and capacitance matrices, respectively [1]. $\boldsymbol{R}_c$ and $\boldsymbol{L}_c$ are symmetric and result from imperfect conductors so that for $(n+1)$ perfect conductors, $\boldsymbol{R}_c = {}_n\boldsymbol{0}_n$ and $\boldsymbol{L}_c = {}_n\boldsymbol{0}_n$ [1]. For $(n+1)$ perfect conductors, $\boldsymbol{G}$, $\boldsymbol{L}$, and $\boldsymbol{C}$ are computed by assuming that the electric and magnetic field intensity vectors lie in planes perpendicular to x and at each frequency satisfy static distributions at each x along the line [1]. In addition, it can be shown that $\boldsymbol{G}$, $\boldsymbol{L}$, and $\boldsymbol{C}$ are symmetric for a homogeneous medium or an inhomogeneous medium [2] and for a lossless medium $\boldsymbol{G} = {}_n\boldsymbol{0}_n$. $\boldsymbol{G}$ and $\boldsymbol{C}$ are of the form $[\boldsymbol{G}]_{ii} = \sum_{j=1}^{n} g_{ij}$, $[\boldsymbol{G}]_{ij} = -g_{ij}$, $[\boldsymbol{C}]_{ii} = \sum_{j=1}^{n} c_{ij}$, $[\boldsymbol{C}]_{ij} = -c_{ij}$ where g_{ii}, c_{ii} and g_{ij}, c_{ij} are the per unit length conductances and capacitances between the ith conductor and the reference conductor and between the ith conductor and the jth conductor, respectively with $i,j = 1,\cdots,n$ [2]. For transmission lines consisting of $(n+1)$ perfect conductors in a homogeneous medium characterized by the scalar permittivity ϵ, permeability μ, and conductivity σ, it may be shown that $\boldsymbol{LC} = \mu\epsilon\boldsymbol{1}_n$ and $\boldsymbol{GL} = \sigma\mu\boldsymbol{1}_n$ where $\boldsymbol{1}_n$ is the $n \times n$ identity matrix with ones on the main diagonal and zeros elsewhere [5].

II. THE MATRIX CHAIN PARAMETERS

Since (1) is a set of first-order ordinary constant-coefficient differential equations in state variable form, the solution is well known [3] and is given by

$$\begin{bmatrix} \boldsymbol{V}(x) \\ \boldsymbol{I}(x) \end{bmatrix} = \boldsymbol{\Phi}(x - x_0) \begin{bmatrix} \boldsymbol{V}(x_0) \\ \boldsymbol{I}(x_0) \end{bmatrix} \tag{3}$$

where the $2n \times 2n$ complex matrix $\boldsymbol{\Phi}(x - x_0)$ is the state transition matrix or chain parameter matrix and x_0 is some arbitrary fixed point along the line with $x \geq x_0$. In addition, the state transition matrix $\boldsymbol{\Phi}(x - x_0)$ has the property $\boldsymbol{\Phi}(0) = \boldsymbol{1}_{2n}$ where $\boldsymbol{1}_{2n}$ is the $2n \times 2n$ identity matrix with $[\boldsymbol{1}_{2n}]_{ii} = 1$ and $[\boldsymbol{1}_{2n}]_{ij} = 0$ for $i,j = 1,\cdots,2n$ and $i \neq j$ [3]. Without loss of generality we may take $x = \mathcal{L}$ and $x_0 = 0$ in (3) resulting in the overall chain parameter matrix of the line $\boldsymbol{\Phi}(\mathcal{L})$. Additionally, it can be shown that the inverse of the state transition matrix or chain parameter matrix is given by $\boldsymbol{\Phi}^{-1}(x - x_0) = \boldsymbol{\Phi}(x_0 - x)$ where the inverse of a matrix $\boldsymbol{M}$ is denoted by $\boldsymbol{M}^{-1}$ [3]. Therefore, $\boldsymbol{\Phi}^{-1}(\mathcal{L}) = \boldsymbol{\Phi}(-\mathcal{L})$.

The chain parameter matrix for uniform lines can be obtained easily since $\boldsymbol{Z}$ and $\boldsymbol{Y}$ in (1) are independent of x. Differentiating the second equation of (1) with respect to x, $\ddot{\boldsymbol{I}}(x) = -\boldsymbol{Y}\dot{\boldsymbol{V}}(x)$, and substituting the first equation of (1) results in

$$\ddot{\boldsymbol{I}}(x) = \boldsymbol{YZI}(x). \tag{4}$$

One may define a change of variables as $\boldsymbol{I}(x) = \boldsymbol{TI}_m(x)$ where $\boldsymbol{T}$ is an $n \times n$ nonsingular complex matrix and $\boldsymbol{I}_m(x)$ is an $n \times 1$ vector of "mode currents." Substituting this in (4) yields

$$\ddot{\boldsymbol{I}}_m(x) = \boldsymbol{T}^{-1}\boldsymbol{YZTI}_m(x). \tag{5}$$

If a similarity transformation $\boldsymbol{T}$ can be found which diagonalizes the matrix product $\boldsymbol{YZ}$ as

$$\boldsymbol{T}^{-1}\boldsymbol{YZT} = \boldsymbol{\gamma}^2 \tag{6}$$

where $\boldsymbol{\gamma}^2$ is an $n \times n$ diagonal matrix with $[\boldsymbol{\gamma}^2]_{ii} = \gamma_i^2$ and $[\boldsymbol{\gamma}^2]_{ij} = 0$ for $i,j = 1,\cdots,n$ and $i \neq j$, then (5) becomes a set of n uncoupled equations and the solution to (4) can be obtained easily as [4]

$$\boldsymbol{I}(x) = \boldsymbol{T}(e^{-\boldsymbol{\gamma}x}\boldsymbol{\alpha}^+ + e^{\boldsymbol{\gamma}x}\boldsymbol{\alpha}^-). \tag{7}$$

Here $e^{\boldsymbol{\gamma}x}$ is an $n \times n$ diagonal matrix with $[e^{\boldsymbol{\gamma}x}]_{ii} = e^{\gamma_i x}$, $[e^{\boldsymbol{\gamma}x}]_{ij} = 0$ for $i,j = 1,\cdots,n$ and $i \neq j$, and $\boldsymbol{\alpha}^+$ and $\boldsymbol{\alpha}^-$ are $n \times 1$ vectors of $2n$ undetermined constants $[\boldsymbol{\alpha}^+]_i = \alpha_i^+$, $[\boldsymbol{\alpha}^-]_i = \alpha_i^-$. It is clear from (5) and (6) that the mode currents $\boldsymbol{I}_m(x)$ consist of n uncoupled waves with propagation constants $\gamma_i, i = 1,\cdots,n$. If the complex scalars γ_i are written as $\gamma_i = \eta_i + j\omega/v_i$, then the attenuation constants and velocities of propagation for each mode become η_i and v_i, respectively. From the second equation of (1), $\boldsymbol{V}(x) = -\boldsymbol{Y}^{-1}\dot{\boldsymbol{I}}(x)$, and (7) we obtain [4]

$$\boldsymbol{V}(x) = \boldsymbol{Y}^{-1}\boldsymbol{T}\boldsymbol{\gamma}\boldsymbol{T}^{-1}\{\boldsymbol{T}(e^{-\boldsymbol{\gamma}x}\boldsymbol{\alpha}^+ - e^{\boldsymbol{\gamma}x}\boldsymbol{\alpha}^-)\} \tag{8}$$

where the square root of $\boldsymbol{\gamma}^2$ is denoted by $\boldsymbol{\gamma}$ with $[\boldsymbol{\gamma}]_{ii} = \gamma_i$ and $[\boldsymbol{\gamma}]_{ij} = 0$ for $i,j = 1,\cdots,n$ and $i \neq j$. Multiplying (7) and (8) by $e^{j\omega t}$, we obtain the voltages and currents in the time domain in terms of forward-traveling waves $\boldsymbol{\mathcal{V}}^+(x,t), \boldsymbol{\mathcal{I}}^+(x,t)$ and backward traveling waves $\boldsymbol{\mathcal{V}}^-(x,t), \boldsymbol{\mathcal{I}}^-(x,t)$ as

$$\boldsymbol{\mathcal{V}}(x,t) = \boldsymbol{\mathcal{V}}^+(x,t) + \boldsymbol{\mathcal{V}}^-(x,t) \tag{9a}$$

and

$$\boldsymbol{\mathcal{I}}(x,t) = \boldsymbol{\mathcal{I}}^+(x,t) - \boldsymbol{\mathcal{I}}^-(x,t). \tag{9b}$$

From (7) and (8) we may identify

$$\mathcal{I}^+(x,t) = Te^{-\gamma x}\alpha^+e^{j\omega t} \tag{10a}$$

$$\mathcal{I}^-(x,t) = -Te^{\gamma x}\alpha^-e^{j\omega t} \tag{10b}$$

$$\mathcal{V}^+(x,t) = Z_C\mathcal{I}^+(x,t) \tag{10c}$$

$$\mathcal{V}^-(x,t) = Z_C\mathcal{I}^-(x,t). \tag{10d}$$

The characteristic impedance matrix Z_C is logically defined from (7), (8), (9), and (10) as

$$Z_C = Y^{-1}T\gamma T^{-1} = ZT\gamma^{-1}T^{-1} \tag{11}$$

and the identity $Y^{-1}T\gamma = ZT\gamma^{-1}$ used in (11) may be easily verified from (6). The square root of YZ may be defined as

$$\sqrt{YZ} = (T\gamma T^{-1}) \tag{12}$$

which is easily verified by forming $YZ = (\sqrt{YZ})(\sqrt{YZ}) = (T\gamma T^{-1})(T\gamma T^{-1}) = T\gamma^2T^{-1}$. Thus (11) may be expressed, symbolically, as

$$Z_C = Y^{-1}(\sqrt{YZ}) = Z(\sqrt{YZ})^{-1} \tag{13}$$

which reduces to a familiar result for the two-conductor case ($n = 1$) where Y and Z become complex scalars. The chain parameter matrix, $\Phi(\mathcal{L})$, may be expressed in partitioned form as

$$\Phi(\mathcal{L}) = \begin{bmatrix} \Phi_{11}(\mathcal{L}) & \Phi_{12}(\mathcal{L}) \\ \Phi_{21}(\mathcal{L}) & \Phi_{22}(\mathcal{L}) \end{bmatrix} \tag{14}$$

where the submatrices $\Phi_{11}(\mathcal{L}), \Phi_{12}(\mathcal{L}), \Phi_{21}(\mathcal{L}), \Phi_{22}(\mathcal{L})$ are $n \times n$ and are obtained by eliminating α^+ and α^- from (7) and (8) to give [4]

$$\Phi_{11}(\mathcal{L}) = 1/2Y^{-1}T(e^{\gamma\mathcal{L}} + e^{-\gamma\mathcal{L}})T^{-1}Y \tag{15a}$$

$$\Phi_{12}(\mathcal{L}) = -1/2Y^{-1}T\gamma(e^{\gamma\mathcal{L}} - e^{-\gamma\mathcal{L}})T^{-1} \tag{15b}$$

$$\Phi_{21}(\mathcal{L}) = -1/2T(e^{\gamma\mathcal{L}} - e^{-\gamma\mathcal{L}})\gamma^{-1}T^{-1}Y \tag{15c}$$

$$\Phi_{22}(\mathcal{L}) = 1/2T(e^{\gamma\mathcal{L}} + e^{-\gamma\mathcal{L}})T^{-1}. \tag{15d}$$

The matrix exponential may be defined as an absolutely convergent matrix infinite series [3]. Therefore, we may form

$$e^{\sqrt{YZ}\mathcal{L}} = 1_n + \sqrt{YZ}\frac{\mathcal{L}}{1!} + (\sqrt{YZ})^2\frac{\mathcal{L}^2}{2!} + (\sqrt{YZ})^3\frac{\mathcal{L}^3}{3!} + \cdots \tag{16a}$$

$$e^{\gamma\mathcal{L}} = 1_n + \gamma\frac{\mathcal{L}}{1!} + \gamma^2\frac{\mathcal{L}^2}{2!} + \gamma^3\frac{\mathcal{L}^3}{3!} + \cdots \tag{16b}$$

$$e^{\sqrt{YZ}\mathcal{L}} = Te^{\gamma\mathcal{L}}T^{-1} \tag{16c}$$

since $\sqrt{YZ} = T\gamma T^{-1}$ [3]. Thus matrix hyperbolic functions may be defined as

$$\begin{aligned}\cosh(\sqrt{YZ}\mathcal{L}) &= 1/2(e^{\sqrt{YZ}\mathcal{L}} + e^{-\sqrt{YZ}\mathcal{L}}) \\ &= 1_n + (\sqrt{YZ})^2\frac{\mathcal{L}^2}{2!} + (\sqrt{YZ})^4\frac{\mathcal{L}^4}{4!} + \cdots \\ &= 1/2T(e^{\gamma\mathcal{L}} + e^{-\gamma\mathcal{L}})T^{-1}\end{aligned} \tag{17a}$$

$$\begin{aligned}\sinh(\sqrt{YZ}\mathcal{L}) &= 1/2(e^{\sqrt{YZ}\mathcal{L}} - e^{-\sqrt{YZ}\mathcal{L}}) \\ &= \sqrt{YZ}\mathcal{L} + (\sqrt{YZ})^3\frac{\mathcal{L}^3}{3!} + (\sqrt{YZ})^5\frac{\mathcal{L}^5}{5!} + \cdots \\ &= 1/2T(e^{\gamma\mathcal{L}} - e^{-\gamma\mathcal{L}})T^{-1}.\end{aligned} \tag{17b}$$

Therefore, using (17), (15) may be expressed symbolically as

$$\Phi_{11}(\mathcal{L}) = Y^{-1}\cosh(\sqrt{YZ}\mathcal{L})Y \tag{18a}$$

$$\begin{aligned}\Phi_{12}(\mathcal{L}) &= -Y^{-1}\sqrt{YZ}\sinh(\sqrt{YZ}\mathcal{L}) \\ &= -Z(\sqrt{YZ})^{-1}\sinh(\sqrt{YZ}\mathcal{L}) \\ &= -Z_C\sinh(\sqrt{YZ}\mathcal{L})\end{aligned} \tag{18b}$$

$$\begin{aligned}\Phi_{21}(\mathcal{L}) &= -\sinh(\sqrt{YZ}\mathcal{L})(\sqrt{YZ})^{-1}Y \\ &= -\sinh(\sqrt{YZ}\mathcal{L})(\sqrt{YZ})Z^{-1} \\ &= -\sinh(\sqrt{YZ}\mathcal{L})Z_C^{-1}\end{aligned} \tag{18c}$$

$$\Phi_{22}(\mathcal{L}) = \cosh(\sqrt{YZ}\mathcal{L}). \tag{18d}$$

Although derived for multiconductor lines, these expressions reduce to a familiar result for the two-conductor line. The order of matrix multiplication is important since the matrix products do not generally commute. For numerical computation one would use the expressions for the submatrices given in (15) since the equivalent symbolic expressions in (18) would be of little value in machine computation.

An equivalent development in terms of the matrix product ZY and $\sqrt{ZY}$ can be obtained. The eigenvalues of YZ, γ_i^2, $i = 1,\cdots,n$, are given by the n roots of [3]

$$\det(\gamma^2 1_n - YZ) = 0 \tag{19}$$

where the determinant of an $n \times n$ matrix M is denoted by det (M). The $n \times 1$ columns of T, T_i where $T = [T_1, T_2, \cdots, T_n]$ are the eigenvectors of YZ and are the solutions to [3]

$$(\gamma_i^2 1_n - YZ)T_i = {}_n0_1. \tag{20}$$

The eigenvalues of ZY can be shown to be the same as the eigenvalues of YZ.[1] This can be shown easily, when Y or Z are nonsingular, by forming $\det(\gamma^2 1_n - YZ) = \det(Y\{\gamma^2 1_n - ZY\}Y^{-1}) = \det(Z^{-1}\cdot\{\gamma^2 1_n - ZY\}Z)$. The determinant of a product of square matrices is equal to the product of their determinants in any order. Thus we may write this as $\det(\gamma^2 1_n - YZ) = \det(Y)\det(Y^{-1})\det(\gamma^2 1_n - ZY) = \det(Z^{-1})\det(Z)\det(\gamma^2 1_n - ZY) = \det(\gamma^2 1_n - ZY)$ since $\det(Y)\det(Y^{-1}) = \det(Z^{-1})\det(Z) = 1$. Also one can form (20) as $Y(\gamma_i^2 1_n - ZY)(Y^{-1}T_i) = {}_n0_1$ so that if Y is nonsingular, then each of the eigenvectors of ZY is equal to the product of Y^{-1} and each of the eigenvectors of YZ (within a scalar constant). Similarly, (20) can be formed as $Z^{-1}(\gamma^2 1_n - ZY)(ZT_i) = {}_n0_1$ so that if Z is nonsingular, then each of the eigenvectors of ZY is equal to the product of Z and each of the eigenvectors of YZ (within a scalar constant). These facts can be used to form the above relations in terms of the matrix product ZY. The order of matrix multiplication is important and $\sqrt{YZ} \neq \sqrt{ZY}$ in general. In fact, one can show that

$$\sqrt{ZY} = Y^{-1}(\sqrt{YZ})Y \tag{21}$$

by forming $\sqrt{ZY}\sqrt{ZY} = (Y^{-1}\sqrt{YZ}Y)(Y^{-1}\sqrt{YZ}Y) = ZY$. Thus the characteristic impedance matrix in terms of $\sqrt{ZY}$ can be expressed symbolically from (13) and (21) as

$$Z_C = (\sqrt{ZY})Y^{-1} = (\sqrt{ZY})^{-1}Z. \tag{22}$$

Additionally, the state transition matrix can be formed as an absolutely convergent matrix infinite series [3]

$$\Phi(\mathcal{L}) = e^{M\mathcal{L}} = 1_{2n} + M\mathcal{L} + M^2\frac{\mathcal{L}^2}{2!} + M^3\frac{\mathcal{L}^3}{3!} + \cdots \tag{23a}$$

where from (1)

$$M = \begin{bmatrix} {}_n0_n & -Z \\ -Y & {}_n0_n \end{bmatrix}. \tag{23b}$$

After obtaining the indicated products in (23) one can identify using (14)

$$\Phi_{11}(\mathcal{L}) = 1_n + ZY\frac{\mathcal{L}^2}{2!} + (ZY)^2\frac{\mathcal{L}^4}{4!} + \cdots \tag{24a}$$

$$\Phi_{12}(\mathcal{L}) = -Z\mathcal{L} - ZYZ\frac{\mathcal{L}^3}{3!} - (ZY)^2Z\frac{\mathcal{L}^5}{5!} - \cdots \tag{24b}$$

$$\Phi_{21}(\mathcal{L}) = -Y\mathcal{L} - YZY\frac{\mathcal{L}^3}{3!} - (YZ)^2Y\frac{\mathcal{L}^5}{5!} - \cdots \tag{24c}$$

[1] See [3, pp. 101–102].

$$\Phi_{22}(\mathcal{L}) = 1_n + YZ\frac{\mathcal{L}^2}{2!} + (YZ)^2\frac{\mathcal{L}^4}{4!} + \cdots. \tag{24d}$$

Matrix hyperbolic functions may logically be defined as absolutely convergent infinite series as in (17) and thus (24) may now be expressed as

$$\begin{aligned}\Phi_{11}(\mathcal{L}) &= \cosh(\sqrt{ZY}\mathcal{L})\\ &= Y^{-1}\left\{1_n + YZ\frac{\mathcal{L}^2}{2!} + (YZ)^2\frac{\mathcal{L}^4}{4!} + \cdots\right\}Y\\ &= Y^{-1}\cosh(\sqrt{YZ}\mathcal{L})Y\end{aligned} \tag{25a}$$

$$\begin{aligned}\Phi_{12}(\mathcal{L}) &= -Z(\sqrt{YZ})^{-1}\left\{\sqrt{YZ}\mathcal{L} + (\sqrt{YZ})^3\frac{\mathcal{L}^3}{3!} + (\sqrt{YZ})^5\frac{\mathcal{L}^5}{5!} + \cdots\right\}\\ &= -Z(\sqrt{YZ})^{-1}\sinh(\sqrt{YZ}\mathcal{L})\end{aligned} \tag{25b}$$

or

$$\begin{aligned}\Phi_{12}(\mathcal{L}) &= -\left\{\sqrt{ZY}\mathcal{L} + (\sqrt{ZY})^3\frac{\mathcal{L}^3}{3!} + (\sqrt{ZY})^5\frac{\mathcal{L}^5}{5!} + \cdots\right\}\sqrt{ZY}Y^{-1}\\ &= -\sinh(\sqrt{ZY}\mathcal{L})\sqrt{ZY}Y^{-1}\end{aligned} \tag{25c}$$

$$\begin{aligned}\Phi_{21}(\mathcal{L}) &= -Y(\sqrt{ZY})^{-1}\left\{\sqrt{ZY}\mathcal{L} + (\sqrt{ZY})^3\frac{\mathcal{L}^3}{3!} + (\sqrt{ZY})^5\frac{\mathcal{L}^5}{5!} + \cdots\right\}\\ &= -Y(\sqrt{ZY})^{-1}\sinh(\sqrt{ZY}\mathcal{L})\end{aligned} \tag{25d}$$

or

$$\begin{aligned}\Phi_{21}(\mathcal{L}) &= -\left\{\sqrt{YZ}\mathcal{L} + (\sqrt{YZ})^3\frac{\mathcal{L}^3}{3!} + (\sqrt{YZ})^5\frac{\mathcal{L}^5}{5!} + \cdots\right\}(\sqrt{YZ})^{-1}Y\\ &= -\sinh(\sqrt{YZ}\mathcal{L})(\sqrt{YZ})^{-1}Y\end{aligned} \tag{25e}$$

$$\begin{aligned}\Phi_{22}(\mathcal{L}) &= \cosh(\sqrt{YZ}\mathcal{L})\\ &= Y\left\{1_n + ZY\frac{\mathcal{L}^2}{2!} + (ZY)^2\frac{\mathcal{L}^4}{4!} + \cdots\right\}Y^{-1}\\ &= Y\cosh(\sqrt{ZY}\mathcal{L})Y^{-1}.\end{aligned} \tag{25f}$$

Recalling the characteristic impedance Z_C from (13) and (22), we may express (25) as

$$\Phi_{11}(\mathcal{L}) = Y^{-1}\cosh(\sqrt{YZ}\mathcal{L})Y = \cosh(\sqrt{ZY}\mathcal{L}) \tag{26a}$$

$$\Phi_{12}(\mathcal{L}) = -Z_C\sinh(\sqrt{YZ}\mathcal{L}) = -\sinh(\sqrt{ZY}\mathcal{L})Z_C \tag{26b}$$

$$\Phi_{21}(\mathcal{L}) = -\sinh(\sqrt{YZ}\mathcal{L})Z_C^{-1} = -Z_C^{-1}\sinh(\sqrt{ZY}\mathcal{L}) \tag{26c}$$

$$\Phi_{22}(\mathcal{L}) = \cosh(\sqrt{YZ}\mathcal{L}) = Y\cosh(\sqrt{ZY}\mathcal{L})Y^{-1} \tag{26d}$$

which of course reduces to a familiar result for the two-conductor case.

III. MATRIX CHAIN PARAMETER IDENTITIES

The purpose of this section is to show some fundamental identities involving the submatrices of the chain parameter matrix Φ_{11}, Φ_{12}, Φ_{21}, and Φ_{22}. The fundamental identities are

Identity 1: $$\Phi_{12}\Phi_{22}\Phi_{12}^{-1}\Phi_{11} - \Phi_{12}\Phi_{21} = 1_n \tag{27a}$$

Identity 2: $$\Phi_{21}\Phi_{11}\Phi_{21}^{-1}\Phi_{22} - \Phi_{21}\Phi_{12} = 1_n \tag{27b}$$

Identity 3: $$\Phi_{12}\Phi_{22}\Phi_{12}^{-1} = \Phi_{11} \tag{27c}$$

Identity 4: $$\Phi_{21}\Phi_{11}\Phi_{21}^{-1} = \Phi_{22} \tag{27d}$$

Identity 5: $$\Phi_{11} = \Phi_{22}^t \tag{27e}$$

where the transpose of a matrix M is denoted by M^t.

Identity 1 and Identity 2 for the two-conductor case reduce to the familar result: $\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21} = 1$, i.e., the determinant of the chain parameter matrix is equal to one, since for $n = 1$, the submatrices become scalars. Similarly, Identities 3,4,5 reduce to a familar result for the two-conductor case, i.e., $\Phi_{11} = \Phi_{22}$. Identities 1,2,3,4 can be readily verified by substituting the form of the submatrices given in (15) and utilizing the fact that $e^{\gamma\mathcal{L}}$, $e^{-\gamma\mathcal{L}}$, and γ are diagonal matrices so that the order of multiplication of these matrices may be interchanged. Identity 5 is easily shown from (24a) and (24d) since the transpose of the sum of any number of matrices is equal to the sum of their transposes and Z and Y are symmetric, i.e., $Z = Z^t$ and $Y = Y^t$.

In addition, both Φ_{12} and Φ_{21} are symmetric matrices since Z and Y are symmetric which can be obviously demonstrated from (24b) and (24c). Also, the following matrix products can be shown to be symmetric: $\Phi_{11}\Phi_{12}, \Phi_{22}\Phi_{21}, \Phi_{12}\Phi_{22}, \Phi_{21}\Phi_{11}$ which can be easily demonstrated from (24). Also from (24b) and (24c) it follows that $Y\Phi_{12} = \Phi_{21}Z$.

The proofs of Identities 1,2,3,4 relied upon the direct substitution of the forms of the chain parameter submatrices in (15) which assumed that YZ is diagonalizable by the similarity transformation T as in (6). The general development for the chain parameter matrix without the assumption of the diagonalizability of YZ is given in [4] in terms of the Jordan canonical form. It is possible to directly show these identities in general regardless of whether or not YZ is diagonalizable by a similarity transformation.

Identities 1,2,3,4 can be directly shown by utilizing the important fact [3]

$$\Phi^{-1}(\mathcal{L}) = \Phi(-\mathcal{L}) \tag{28a}$$

or

$$\Phi(\mathcal{L})\Phi(-\mathcal{L}) = 1_{2n}. \tag{28b}$$

This relationship for the inverse of the state transition or chain parameter matrix follows from (3) and holds, in general, for any system of first-order ordinary differential equations and does not depend on the structure of M in (23b) [3]. Forming this relation gives

$$\begin{bmatrix}\Phi_{11}(\mathcal{L}) & \Phi_{12}(\mathcal{L})\\ \Phi_{21}(\mathcal{L}) & \Phi_{22}(\mathcal{L})\end{bmatrix}\begin{bmatrix}\Phi_{11}(-\mathcal{L}) & \Phi_{12}(-\mathcal{L})\\ \Phi_{21}(-\mathcal{L}) & \Phi_{22}(-\mathcal{L})\end{bmatrix} = \begin{bmatrix}1_n & {}_n0_n\\ {}_n0_n & 1_n\end{bmatrix} \tag{29}$$

which yields the identities

$$\Phi_{11}(\mathcal{L})\Phi_{11}(-\mathcal{L}) + \Phi_{12}(\mathcal{L})\Phi_{21}(-\mathcal{L}) = 1_n \tag{30a}$$

$$\Phi_{22}(\mathcal{L})\Phi_{22}(-\mathcal{L}) + \Phi_{21}(\mathcal{L})\Phi_{12}(-\mathcal{L}) = 1_n \tag{30b}$$

$$\Phi_{11}(\mathcal{L})\Phi_{12}(-\mathcal{L}) + \Phi_{12}(\mathcal{L})\Phi_{22}(-\mathcal{L}) = {}_n0_n \tag{30c}$$

$$\Phi_{21}(\mathcal{L})\Phi_{11}(-\mathcal{L}) + \Phi_{22}(\mathcal{L})\Phi_{21}(-\mathcal{L}) = {}_n0_n. \tag{30d}$$

By utilizing the specific structure of the coefficient matrix in (1) and the resulting series expansion of the state transition matrix in (24), it is clear that

$$\Phi_{11}(-\mathcal{L}) = \Phi_{11}(\mathcal{L}) \tag{31a}$$

$$\Phi_{22}(-\mathcal{L}) = \Phi_{22}(\mathcal{L}) \tag{31b}$$

$$\Phi_{12}(-\mathcal{L}) = -\Phi_{12}(\mathcal{L}) \tag{31c}$$

$$\Phi_{21}(-\mathcal{L}) = -\Phi_{21}(\mathcal{L}). \tag{31d}$$

Substituting the relations in (31) into (30), we obtain

$$\Phi_{11}(\mathcal{L})\Phi_{11}(\mathcal{L}) - \Phi_{12}(\mathcal{L})\Phi_{21}(\mathcal{L}) = 1_n \tag{32a}$$

$$\Phi_{22}(\mathcal{L})\Phi_{22}(\mathcal{L}) - \Phi_{21}(\mathcal{L})\Phi_{12}(\mathcal{L}) = 1_n \tag{32b}$$

$$-\Phi_{11}(\mathcal{L})\Phi_{12}(\mathcal{L}) + \Phi_{12}(\mathcal{L})\Phi_{22}(\mathcal{L}) = {}_n0_n \tag{32c}$$

$$\Phi_{21}(\mathcal{L})\Phi_{11}(\mathcal{L}) - \Phi_{22}(\mathcal{L})\Phi_{21}(\mathcal{L}) = {}_n0_n. \tag{32d}$$

If $\Phi_{12}(\mathcal{L})$ and $\Phi_{21}(\mathcal{L})$ are nonsingular, then (32c) becomes Identity 3 and (32d) becomes Identity 4. Substituting these two identities into (32a) and (32b) we obtain Identity 1 and Identity 2, respectively. Note that Identities 1,2,3,4 do not rely on Z and Y being symmetric.

IV. INCORPORATING THE TERMINATION NETWORKS

The desired end result in an analysis of transmission line behavior is a determination of the voltages and currents along the line when the line is driven by linear termination networks at the ends of the line. For these purposes, we choose to characterize these linear termination networks by "generalized Thevenin equivalents" as

$$V(0) = E_0 - Z_0 I(0) \tag{33a}$$

$$V(\mathcal{L}) = E_{\mathcal{L}} + Z_{\mathcal{L}} I(\mathcal{L}) \tag{33b}$$

where E_0 and $E_{\mathcal{L}}$ are $n \times 1$ complex vectors of equivalent open circuit port excitation voltages (with respect to the reference conductor) and Z_0 and $Z_{\mathcal{L}}$ are $n \times n$ complex symmetric impedance matrices. The matrix chain parameters relate the voltages and currents at the ends of the line as

$$V(\mathcal{L}) = \Phi_{11}(\mathcal{L})V(0) + \Phi_{12}(\mathcal{L})I(0) \tag{34a}$$

$$I(\mathcal{L}) = \Phi_{21}(\mathcal{L})V(0) + \Phi_{22}(\mathcal{L})I(0). \tag{34b}$$

The objective now is to eliminate $V(0)$ and $V(\mathcal{L})$ from (33) and (34) to yield a set of $2n$ equations in the $2n$ unknowns $I(0)$ and $I(\mathcal{L})$. Substituting (33a) into (34b) and rearranging yields the first equation

$$I(\mathcal{L}) + (\Phi_{21}Z_0 - \Phi_{22})I(0) = \Phi_{21}E_0. \tag{35}$$

The remaining equation can be obtained by substituting (33b) into (34a) to yield

$$E_{\mathcal{L}} + Z_{\mathcal{L}}I(\mathcal{L}) = \Phi_{11}V(0) + \Phi_{12}I(0). \tag{36}$$

Substituting from (34b)

$$V(0) = \Phi_{21}^{-1}I(\mathcal{L}) - \Phi_{21}^{-1}\Phi_{22}I(0) \tag{37}$$

into (36), rearranging and multiplying on the left by Φ_{21} yields

$$(-\Phi_{21}Z_{\mathcal{L}} + \Phi_{21}\Phi_{11}\Phi_{21}^{-1})I(\mathcal{L}) + (\Phi_{21}\Phi_{12} - \Phi_{21}\Phi_{11}\Phi_{21}^{-1}\Phi_{22})I(0) = \Phi_{21}E_{\mathcal{L}}. \tag{38}$$

Using Identity 2 and Identity 4 in (38) yields

$$(\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})I(\mathcal{L}) + I(0) = -\Phi_{21}E_{\mathcal{L}}. \tag{39}$$

Equations (35) and (39) can be arranged in matrix form as

$$\begin{bmatrix} (\Phi_{21}Z_0 - \Phi_{22}) & 1_n \\ 1_n & (\Phi_{21}Z_{\mathcal{L}} - \Phi_{22}) \end{bmatrix} \begin{bmatrix} I(0) \\ I(\mathcal{L}) \end{bmatrix} = \begin{bmatrix} \Phi_{21} & E_0 \\ -\Phi_{21} & E_{\mathcal{L}} \end{bmatrix} \tag{40}$$

which has a highly sparse coefficient matrix with $2(n^2 - n)$ of the total $4n^2$ elements identically zero. Equation (40) can also be solved explicitly for $I(0)$ and $I(\mathcal{L})$ as

$$\{1_n - (\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})(\Phi_{21}Z_0 - \Phi_{22})\}I(0) = -\Phi_{21}E_{\mathcal{L}} - (\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})\Phi_{21}E_0 \tag{41a}$$

$$I(\mathcal{L}) = -(\Phi_{21}Z_0 - \Phi_{22})I(0) + \Phi_{21}E_0. \tag{41b}$$

$V(x)$ and $I(x)$ at any point along the line can be found from (3) once $I(0)$ is obtained from the solution of (40) or (41a) and $V(0)$ is obtained from (33a).

The identities have reduced the number of redundant matrix multiplications and, moreover, only two of the four matrix chain parameter submatrices are required to be computed; Φ_{21} and Φ_{22} [see (15)]. Reducing the number of required matrix multiplications is an important consideration in numerical machine computation. For example, n^3 operations (multiplications or divisions) are required to multiply two "full" $n \times n$ matrices which is the minimum number of operations required to invert a "full" $n \times n$ matrix [6]. The solution of a set of n equations in n unknowns by Gauss elimination requires $n^3/3 + n^2 - n/3$ operations or $n^3/3$ for large n [6]. Forming $\Phi_{21}Z_0$ and $\Phi_{21}Z_{\mathcal{L}}$ require $2n^3$ operations. Therefore, solution of (40) requires on the order of $(2n)^3/3 + 2n^3 = 14n^3/3$ total operations (neglecting the n^2 operations required to form $\Phi_{21}E_0$ and $\Phi_{21}E_{\mathcal{L}}$). Solution of (41a) requires an additional n^3 operations for the multiplication of $(\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})(\Phi_{21}Z_0 - \Phi_{22})$ and n^3 operations for the multiplication of $(\Phi_{21}Z_{\mathcal{L}} - \Phi_{22})\Phi_{21}$. Thus the total number of operations required to solve (41) is on the order of $n^3/3 + 4n^3 = 13n^3/3$ operations. Therefore, it may be more efficient to solve (40) rather than (41) since almost the same number of operations are involved and the sparsity of (40) can be used to reduce the required number of operations even further.

V. SUMMARY

Certain matrix identities among the submatrices of the chain parameter matrix for multiconductor transmission lines have been shown. The identities reduce to familiar results for the two-conductor transmission line where the submatrices become complex scalars. The order of multiplication of the submatrices must be carefully adhered to for the multiconductor case since the various matrix products do not in general commute. A set of matrix equations which incorporate the termination networks for the total solution of the line currents was formulated. Using the matrix identities the coefficient matrix was reduced to a highly sparse and efficient form.

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Green's Function in a Region with Inhomogeneous, Isotropic Dielectric Media

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Abstract—**The reciprocity relation satisfied by the Green's function for the inhomogeneous partial differential equation in a multidielectric region with inhomogeneous, isotropic media is derived by using Green's theorem.**

I. INTRODUCTION

The calculation of the parameters of a microstrip line based on a TEM approximation is useful for the design of microwave integrated circuit structures. The parameters often can be calculated by variational techniques using Green's functions [1]–[5]; however, the Green's function satisfying the boundary conditions must be obtained first.

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